PROBLEM No. 1 / I

Define the following numbers:

$ULP = 2^{-23}$; $EM = 2^{-24}$;   and EPS = `EPSILON(real*4)`,

where `EPSILON(x)` is the intrinsic Fortran function.

Write a program which computes in *single precision*, and displays the values:

1) ULP; EM; EPS;

2) $u = 1.0 + ULP$; $u1 = 1.0 + EM$; $u2 = 1.0 + EPS$;

Check (in the program) if u, $u1$, and $u2$ are greater than or equal to 1.0.

Explain the results.

PROBLEM No. 2 / I

Calculate the values of functions *f*, *f2*, and *g*, at the following *x* values:

$x = 10^i$, $i = 1, 2, \ldots, 7$.

$f(x) = x(\sqrt{x+2} - \sqrt{x-1})$ ... In single precision;

$f2(x) = x(\sqrt{x+2} - \sqrt{x-1})$ … In double precision;

$g(x) = \dfrac{3x}{\sqrt{x+2} + \sqrt{x+1}}$ … In single precision.

- Tabulate computed values ($f2(x)$ with 15 significant digits), and explain the results.
- Find out the number of correct significant digits of $f(x)$ value, for $x = 10^6$.

PROBLEM No. 3 / I

Equation $f(x) = e^x - 4x^2 = 0$ has three roots.

Find the roots, with tolerance $EPS = 10^{-6}$, by:

- BISECTION Method
- NEWTON Method

Compare the number of iterations, needed to meet tolerance EPS.

PROBLEM No. 4 / I

Consider the equation $f(x) = 0$, where:

$$f(x) = 0.98\cos(x) - x + 1.58$$

1) Solve the equation by NEWTON method, picking $x_0 = 1.5$ and tolerance $EPS = 10^{-6}$.
2) Solve again by SECANT method, with $x_0 = 1.5$, $x_1 = 1.6$, and $EPS = 10^{-6}$.
3) Compare the number of iterations in the two methods.

PROBLEM No. 5 / I

Given the equation

$$tg(x) = \frac{8 - x}{x + 0.2}.$$

1) Solve the equation by NEWTON method, choosing $x_0 = 4$ and tolerance

   $EPS = 10^{-6}$.

2) Solve the equation by SECANT method, with $x_0 = 3.5$, $x_1 = 4$, and

   $EPS = 10^{-6}$.

3) Compare the number of iterations in the two methods.

PROBLEM No. 6 / I

Consider the function

$$f(x) = x + e^{-px^2} \cos(x),$$

where p is a parameter.

Equation $f(x) = 0$ has a unique root in the interval $(-1, 0)$.

1) Find the roots for the following p values: $p = 1;\quad 5;$ and 25, with a tolerance

   $EPS = 10^{-6}$.

2) Solve by Newton method the case $p = 25$, with $EPS = 10^{-6}$, and $x_0 = 0$.

3) Comment the result.

PROBLEM No. 7 / I

Let f be the friction factor for the flow of a suspension, R the Reynolds number, and k a constant depending of the suspension concentration. These quantities are related by the empirical relation (Lee & Duffy, 1976):

$$\frac{1}{\sqrt{f}} = \frac{1}{k}\ln(R\sqrt{f}) + 14 - \frac{5.6}{k}$$

Determine f for the values: $k = 0.28$ and $R = 3750$.

PROBLEM No. 8 / I

The equation $e^x - 5x^2 = 0$ has a root in the interval $[4.5, 5]$.

Consider the equation put in the form $x = g(x)$, where:

1) $g(x) = \sqrt{e^x / 5}$

Iterate (in the fixed-point method) with $x_0 = 4.7$. The iteration does not converge. Why?

2) $g(x) = x - m(e^x - 5x^2)$

- Determinate m such that the fixed-point iteration converge, and compute the root.
- Find out the value of m, for which the iteration converges the most rapidly.

PROBLEM No. 9 / I

Consider the equation $x = g(x)$, where

$g(x) = 1.6 + 0.98\cos(x)$

1) Iterate in single precision with $x_0 = 1.58$, tolerance $EPS = 10^{-6}$, and limited number of iterations $NLIM \geq 500$. Comment the result.
2) Repeat the iteration in double precision, and find the root with tolerance $EPS = 10^{-9}$.

PROBLEM No. 10 / I

Determine t, with a tolerance of 10^{-5}, from the equation

$e^{-t/2}\cosh^{-1}(e^{t/2}) = \sqrt{0.5L_{cr}}$,

for $L_{cr} = 0.088$. (t is the temperature in the interior of a material with imbedded heat sources; Frank-Kamenetski, 1955).

PROBLEM No. 11 / I

Given the LAGUERRE polynomial of order 5:

$$L_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)$$

1) Find initial approximations of the roots (by algebraic methods, function graph, etc.)
2) Calculate the roots, with tolerance 10^{-6}.

PROBLEM No. 12 / I

Given the LEGENDRE polynomial of order 6

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5),$$

1) Find initial approximations of the roots (by algebraic methods, function graph, etc.)
2) Calculate the roots, with tolerance 10^{-6}.

Note: All roots are of modulus < 1.

PROBLEM No. 13 / I

The polynomial

$$p(x) = x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120$$

has the roots: $x_1 = 1; x_2 = 2; \ldots; x_5 = 5$.

Let $\tilde{p}(x)$ be the polynomial obtained form $p(x)$ by replacing the coefficient

$a_4 = -15$ of x^4, with $\tilde{a}_4 = -15.003$.

- Calculate the roots of $\tilde{p}(x)$.
- Calculate the modulus of the ratio: relative perturbation in the root x_5 / relative perturbation in the coefficient a_4. Comment the result.

  (If a perturbed becomes $\tilde{a}$, the relative perturbation is: $(\tilde{a} - a)/a$.)

PROBLEM No. 14 / I

The polynomial

$$p(x) = x^5 - 18x^4 + 118x^3 - 348x^2 + 457x - 210$$

Has the roots: $x_1 = 1; x_2 = 2; x_3 = 3; x_4 = 5; x_5 = 7$.

Let $\tilde{p}(x)$ be the polynomial obtained form $p(x)$ by replacing the coefficient

$a_3 = 118$ of x^3, with $\tilde{a}_3 = 118.02$.

- Calculate the roots of $\tilde{p}(x)$.
- Calculate the ratio: maximum relative perturbation in the roots (in modulus) / the relative perturbation in the coefficient a_3. Comment the result.

  (If a perturbed becomes $\tilde{a}$, the relative perturbation is: $(\tilde{a} - a)/a$.)

PROBLEM No. 15 / I

Consider the system:

$$\begin{cases} xy - z^2 = 2 \\ -xyz - x^2 + y^2 = 4 \\ e^x - e^y - z = 7 \end{cases}$$

Solve by NEWTON method, with tolerance $EPS = 10^{-6}$.

1) With initial approximation $w_0 = (1, 1, 1)$

2) With initial approximation $w_0 = (2, 2, -1)$

Compare the number of iterations and explain.

PROBLEM No. 16 / I

Consider the system:

$$\begin{cases} x - y + \sqrt{x + y} = 1 \\ x + y - e^{x-y} = 3 \end{cases}$$

Solve the system, by iteration with constant matrix A (updated after 3 steps), with tolerance $EPS = 10^{-6}$ and initial approximation $w_0 = (1, 2)$.

PROBLEM No. 17 / I

Consider the system:

$$\begin{cases} x^2 + x - y^2 = 1 \\ y - \sin(x)^2 = 0 \end{cases}$$

Solve the system, with tolerance $EPS = 10^{-6}$, and initial approximations:

$w_0^{(1)} = (0.7, 0.5)$; $w_0^{(2)} = (-1.5, 0.4)$:

1) by NEWTON method;
2) by iteration with constant matrix A (with updating after 3 steps).

PROBLEM No. 18 / I

Consider the system:

$$\begin{cases} x^3 + 3y^2 = 21 \\ x^2 + 2y = -2 \end{cases}$$

- Find the initial approximations x_0, y_0, from the intersection of the two graphs.
- Solve by NEWTON method, with tolerance $EPS = 10^{-6}$.

PROBLEM No. 19 / I

Consider the system:

$$\begin{cases} \sin(xy) = 0.5 \\ \cos(x) = e^{y} \end{cases}$$

- Solve by NEWTON method, with tolerance $EPS = 10^{-6}$: Find the roots near $w_0 = (-1, -0.5)$, and $w_0 = (5, -1)$, respectively
- Solve again, by iteration with constant matrix A (with updating after 3 steps).

PROBLEM No. 20 / I

Consider the system:

$$\begin{cases} f_1(x, y) = x^2 + 4y^2 - 9 = 0 \\ f_2(x, y) = -14x^2 + 18y + 45 = 0 \end{cases}$$

- Determine the initial approximations x_0, y_0, from the intersection of curves $f_1(x, y) = 0$ and $f_2(x, y) = 0$.
- Solve by NEWTON method, with tolerance $EPS = 10^{-6}$.

PROBLEM No. 21 / I

Consider the system:

$$\begin{cases} x^2 + y^2 + z^2 = 6.4 \\ xyz = -2.2 \\ x + y - z^2 = 2.4 \end{cases}$$

Solve by NEWTON method, with tolerance $EPS = 10^{-6}$ and initial approximation $w_0 = (2., 1, -1)$.

PROBLEM No. 22 / I

Consider the system:

$$\begin{cases} x^2 + 2\sin(y) + z = -0.1 \\ \cos(y) - z = 2.1 \\ x^2 + y^2 + z^2 = 2 \end{cases}$$

Find the solution near $w_0 = (1, 0, -1)$, with tolerance $EPS = 10^{-6}$, either by Newton method, or by iteration with constant matrix A (with updating after 3 steps).

PROBLEM No. 23 / I

Consider the system:

$$\begin{cases} (x-y)(x+y)^{1/2} = 3 \\ x - \log(x-y) = 1 \end{cases}$$

Solve the system, with:

- Tolerance $EPS = 10^{-6}$.
- Initial approximation $w_0 = (2, -0.5)$.

Use either iteration with constant matrix *A* (updated after 3 steps), or Newton method.

PROBLEM No. 24 / I

In the problem of missile interception, the following system is obtained:

$$\begin{cases} t\cos(\alpha) + t - 1 = 0 \\ t\sin(\alpha) - 0.1t^2 + e^{-t} - 1 = 0 \end{cases}$$

(*t* is the time, and α is the firing angle of the interceptor.)

Solve the system, with tolerance $EPS = 10^{-6}$ and initial approximation $w_0 = (0.5, 1)$, by:

1) Iteration with constant matrix *A* (with updating after 3 steps)
2) NEWTON method.

PROBLEM No. 25 / I

Given the linear system with matrix:

$$A = \begin{bmatrix} 3.01 & 6.03 & 1.99 \\ 1.27 & 4.16 & -1.23 \\ .987 & -4.81 & 9.34 \end{bmatrix}$$

1) Calculate the solution for the right-hand side $[1 \quad 1 \quad 1]^T$.
2) Repeat part (1), with matrix A' obtained from A, by the replacements: $3.01 \to 3.00$ (element a_{11}) and $.987 \to .990$ (element a_{31}). Compare the results and conclude about the conditioning of the problem.

PROBLEM No. 26 / I

Given the matrix:

$$A = \begin{bmatrix} 3.01 & 6.03 & 1.99 \\ 1.27 & 4.16 & -1.23 \\ .987 & -4.81 & 9.34 \end{bmatrix}.$$

1) Calculate the inverse A^{-1} of A.
2) Calculate the following number: $cond(A)_\infty = \| \mathbf{A} \|_\infty \cdot \| \mathbf{A}^{-1} \|_\infty$.

   (*cond(A)* is called number of condition)

PROBLEM No. 27 / I

Given the matrix

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.501 \\ 0.3 & 0.4 & 0.5 & 0.6 \\ 0.4 & 0.5 & 0.6 & 0.7 \\ 0.501 & 0.6 & 0.7 & 0.8 \end{bmatrix}$$

1) Calculate the inverse A^{-1} of *A*.

2) Calculate the following number: $cond(A)_1 = \| \mathbf{A} \|_1 \cdot \| \mathbf{A}^{-1} \|_1$.

   (*cond(A)* is called number of condition)

PROBLEM No. 28 / II

Given the HILBERT matrix of order 4:

$$H_4 = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$$

1) Solve the linear system $H_4 x = b$, for:

   $b = [1 \quad 0.2 \quad 0.3 \quad 0.4]^T$, and $b = [1.02 \quad 0.2 \quad 0.3 \quad 0.4]^T$.

2) Calculate the ratio: the maximum relative perturbation (in modulus) in solution / the relative perturbation in the right-hand side b_1 (in modulus).

   Comment the result.

   (If *a* perturbed becomes $\tilde{a}$, the relative perturbation is: $(\tilde{a} - a)/a$.)

PROBLEM No. 29 / I

Consider the HILBERT matrix of order 5:

$$H_5 = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix};$$

1) Solve the linear system $H_5 x = b$, for:

   $b = [1.0 \quad 0.6 \quad 0.4 \quad 0.3 \quad 0.3]^T$, and $\tilde{b} = [1.02 \quad 0.6 \quad 0.4 \quad 0.3 \quad 0.3]^T$.

2) Compute the ratio: maximum relative perturbation in solution (in modulus) / relative perturbation in right-hand side b_1 (in modulus). Comment the result.

   (If a perturbed becomes $\tilde{a}$, the relative perturbation is: $(\tilde{a} - a)/a$.)

PROBLEM No. 30 / I

Consider the linear system $Ax = b$, where:

$$A = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix}, \text{ and } b = \left[\begin{array}{c:c} 1 & 1 \\ -2 & 0 \\ -2 & 1 \\ -2 & 0 \\ 1 & 1 \end{array}\right].$$

1) Solve the system by LU method.
2) How can be computed the determinant of A?

PROBLEM No. 31 / I

Consider the linear system $Ax = b$, where:

$$A = \begin{bmatrix} 16 & 4 & 8 \\ 4 & 5 & -4 \\ 8 & -4 & 22 \end{bmatrix}, \text{ and } b = \left[\begin{array}{c:c:c} 1 & 2 & 28 \\ 1 & 1 & 5 \\ 1 & -1 & 26 \end{array}\right].$$

1) Solve the system by LU method.

2) Compute the determinant of A.

PROBLEM No. 32 / I

Consider the HILBERT matrix of order 3:

$$H_3 = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix};$$

Matrix entries are: $h_{ij} = 1./(i + j - 1)$.

1) Calculate H_3^{-1} in single precision.

2) Calculate H_3^{-1} in double precision.

3) Calculate in single precision the *analytical* inverse $\left(H_3^{-1}\right)_T = [\alpha_{ij}]$, where:

$$\alpha_{ij} = (-1)^{i+j} \frac{(n+i-1)!(n+j-1)!}{(i+j-1)[(i-1)!(j-1)!]^2 (n-i)!(n-j)!}$$

Compare the elements of the inverses in part (1) and (2), with those of the inverse in part (3). Comment the comparison results.

PROBLEM No. 33 / I

Consider the following linear system $Ax = b$:

$$A = \begin{bmatrix} 6 & 1 & 1 & 0 & 0 \\ 1 & 6 & 1 & 1 & 0 \\ 1 & 1 & 6 & 1 & 1 \\ 0 & 1 & 1 & 6 & 1 \\ 0 & 0 & 1 & 1 & 6 \end{bmatrix}; \qquad b = \left[\begin{array}{c:c} 1 & 0.8 \\ 1 & 0.9 \\ 1 & 1. \\ 1 & 0.9 \\ 1 & 0.8 \end{array}\right];$$

Matrix A is positive definite.

1) Solve the system by CHOLESKY Method.
2) Print the matrix L, and calculate the determinant of A.

PROBLEM No. 34 / I

Given the linear system $Ax = b$, where:

$$A = \begin{bmatrix} 6 & -36 & 12 \\ -36 & 218 & -74 \\ 12 & -74 & 64 \end{bmatrix}; \qquad b = \begin{bmatrix} 1 & -1.8 \\ 1 & 10.8 \\ 1 & 0.2 \end{bmatrix};$$

Matrix A is positive definite.

3) Solve the system by CHOLESKY Method.
4) Print the matrix L, and calculate the determinant of matrix A.