PROBLEM No. 1 / I

Define the following numbers:

$ULP = 2^{-23}$; $EM = 2^{-24}$;   and $EPS =$ `EPSILON(real*4)`,

where `EPSILON(x)` is the intrinsic Fortran function.

Write a program which computes in *single precision*, and displays the values:

1) ULP; EM; EPS;

2) $u = 1.0 + ULP$; $u1 = 1.0 + EM$; $u2 = 1.0 + EPS$;

Check (in the program) if u, $u1$, and $u2$ are greater than or equal to 1.0.

Explain the results.

PROBLEM No. 2 / I

Calculate the values of functions f, $f2$, and g, at the following x values:

$x = 10^i$, $i = 1, 2, \ldots, 7$.

$f(x) = x(\sqrt{x+2} - \sqrt{x-1})$ ... In single precision;

$f2(x) = x(\sqrt{x+2} - \sqrt{x-1})$ … In double precision;

$g(x) = \dfrac{3x}{\sqrt{x+2} + \sqrt{x+1}}$ … In single precision.

- Tabulate computed values ($f2(x)$ with 15 significant digits), and explain the results.
- Find out the number of correct significant digits of $f(x)$ value, for $x = 10^6$.

PROBLEM No. 3 / I

Equation $f(x) = e^x - 4x^2 = 0$ has three roots.

Find the roots, with tolerance $EPS = 10^{-6}$, by:

- BISECTION Method
- NEWTON Method

Compare the number of iterations, needed to meet tolerance EPS.

PROBLEM No. 4 / I

Consider the equation $f(x) = 0$, where:

$$f(x) = 0.98\cos(x) - x + 1.58$$

1) Solve the equation by NEWTON method, picking $x_0 = 1.5$ and tolerance $EPS = 10^{-6}$.
2) Solve again by SECANT method, with $x_0 = 1.5$, $x_1 = 1.6$, and $EPS = 10^{-6}$.
3) Compare the number of iterations in the two methods.

PROBLEM No. 5 / I

Given the equation

$$tg(x) = \frac{8 - x}{x + 0.2}.$$

1) Solve the equation for the root near $x_0 = 4$, with tolerance $EPS = 10^{-6}$ by NEWTON method and, by SECANT method.

2) Compare the number of iterations in the two methods.

PROBLEM No. 6 / I

Consider the function

$$f(x) = x + e^{-px^2} \cos(x),$$

where p is a parameter.

Equation $f(x) = 0$ has a unique root in the interval $(-1, 0)$.

1) Find the roots for the following p values: $p = 1; \quad 5;$ and 25, with a tolerance $EPS = 10^{-6}$.

2) Solve by Newton method the case $p = 25$, with $EPS = 10^{-6}$, and $x_0 = 0$.

3) Comment the result.

PROBLEM No. 7 / I

From the equation bellow, determine *f* with a tolerance of 10^{-6}, for the following values: $k = 0.28$ and $R = 3750$.

$$\frac{1}{\sqrt{f}} = \frac{1}{k}\ln(R\sqrt{f}) + 14 - \frac{5.6}{k}$$

(Empirical relation, Lee & Duffy (1976): *f* is the friction factor for the flow of a suspension, *R* the Reynolds number, and *k* a constant depending of the suspension concentration.)

PROBLEM No. 8 / I

The equation $e^x - 5x^2 = 0$ has a root in the interval $[4.5, 5]$.

Consider the equation put in the form $x = g(x)$, where:

1) $g(x) = \sqrt{e^x / 5}$

Iterate (in the fixed-point method) with $x_0 = 4.7$. The iteration does not converge. Why?

2) $g(x) = x - m(e^x - 5x^2)$

- Determinate *m* such that the fixed-point iteration converge, and compute the root.
- Find out the value of *m*, for which the iteration converges the most rapidly.

PROBLEM No. 9 / I

Consider the equation $x = g(x)$, where

$$g(x) = \frac{2.1 + 0.2tg(x)}{1 + tg(x)}$$

1) Iterate in single precision with $x_0 = 1.$, tolerance $EPS = 10^{-6}$, şi limited number of iterations $NLIM \geq 2000$.
2) Repeat the iteration in double precision and fine the root with tolerance $EPS = 10^{-9}$.
3) Explain why a large number of iterations is needed, before meeting an iteration halting criterion.

PROBLEM No. 10 / I

Determine t, with a tolerance of 10^{-5}, from the equation

$$e^{-t/2} \cosh^{-1}(e^{t/2}) = \sqrt{0.5L_{cr}},$$

for $L_{cr} = 0.088$. (t is the temperature in the interior of a material with imbedded heat sources; Frank-Kamenetski, 1955).

PROBLEM No. 11 / I

Given the LAGUERRE polynomial of order 5:

$$L_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)$$

1) Find initial approximations of the roots (by algebraic methods, function graph, etc.)
2) Calculate the roots, with tolerance 10^{-6}.

PROBLEM No. 12 / I

Given the LEGENDRE polynomial of order 6

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5),$$

1) Find initial approximations of the roots (by algebraic methods, function graph, etc.)
2) Calculate the roots, with tolerance 10^{-6}.

Note: All roots are of modulus < 1.

PROBLEM No. 13 / I

The polynomial

$$p(x) = x^6 - 21x^5 + 175x^4 - 735x^3 + 1624x^2 - 1764x + 720$$

has the roots: $x_i = i,\ i = \overline{1,6}$.

Let $\tilde{p}(x)$ be the polynomial obtained form $p(x)$ by replacing the coefficient

$a_5 = -21$ of x^5, with $\tilde{a}_5 = -21.002$.

- Calculate the roots of $\tilde{p}(x)$.
- Calculate the ratio: largest relative perturbation (in modulus) in the roots / relative perturbation (in modulus) in coefficient a_5. Comment the result.

  (If a perturbed becomes $\tilde{a}$, the relative perturbation is: $(\tilde{a} - a)/a$.)

PROBLEM No. 14 / I

Consider the system:

$$\begin{cases} xy - z^2 = 2 \\ -xyz - x^2 + y^2 = 4 \\ e^x - e^y - z = 7 \end{cases}$$

Solve by NEWTON method, with tolerance $EPS = 10^{-6}$.

1) With initial approximation $w_0 = (1, 1, 1)$

2) With initial approximation $w_0 = (2, 2, -1)$

Compare the number of iterations and explain.

PROBLEM No. 15 / I

Consider the system:

$$\begin{cases} x - y + \sqrt{x+y} = 1 \\ x + y - e^{x-y} = 3 \end{cases}$$

Solve the system, by iteration with constant matrix A (updated after 3 steps), with tolerance $EPS = 10^{-6}$ and initial approximation $w_0 = (1, 2)$.

PROBLEM No. 16 / I

Consider the system:

$$\begin{cases} x^2 + x - y^2 = 1 \\ y - \sin(x)^2 = 0 \end{cases}$$

Solve the system, with tolerance $EPS = 10^{-6}$, and initial approximations:

$w_0^{(1)} = (0.7, 0.5)$; $w_0^{(2)} = (-1.5, 0.4)$:

1) by NEWTON method;

2) by iteration with constant matrix A (with updating after 3 steps).

PROBLEM No. 17 / I

Consider the system:

$$\begin{cases} x^3 + 3y^2 = 21 \\ x^2 + 2y = -2 \end{cases}$$

- Find the initial approximations x_0, y_0, from the intersection of the two graphs.
- Solve by NEWTON method, with tolerance $EPS = 10^{-6}$.

PROBLEM No. 18 / I

Consider the system:

$$\begin{cases} \sin(xy) = 0.5 \\ \cos(x) = e^y \end{cases}$$

- Solve by NEWTON method, with tolerance $EPS = 10^{-6}$: Find the roots near

  $w_0 = (-1, -0.5)$, and $w_0 = (5, -1)$, respectively;
- Solve again, by iteration with constant matrix A (with updating after 3 steps).

PROBLEM No. 19 / I

Consider the system:

$$\begin{cases} x^2 + y^2 + z^2 = 6.4 \\ xyz = -2.2 \\ x + y - z^2 = 2.4 \end{cases}$$

Solve by NEWTON method, with tolerance $EPS = 10^{-6}$ and initial approximation

$w_0 = (2., 1, -1)$.

PROBLEM No. 20 / I

Given the linear system with matrix:

$$A = \begin{bmatrix} 3.01 & 6.03 & 1.99 \\ 1.27 & 4.16 & -1.23 \\ .987 & -4.81 & 9.34 \end{bmatrix}$$

1) Calculate the solution for the right-hand side $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$.

2) Repeat part (1), with matrix A' obtained from A, by the replacements: $3.01 \rightarrow 3.00$ (element a_{11}) and $.987 \rightarrow .990$ (element a_{31}). Compare the results and conclude about the conditioning of the problem.

PROBLEM No. 21 / I

Given the matrix:

$$A = \begin{bmatrix} 3.01 & 6.03 & 1.99 \\ 1.27 & 4.16 & -1.23 \\ .987 & -4.81 & 9.34 \end{bmatrix}.$$

1) Calculate the inverse A^{-1} of A.

2) Calculate the condition number: $cond(A)_\infty$. Is A well- or ill-conditioned?

PROBLEM No. 22 / I

Given the matrix

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.501 \\ 0.3 & 0.4 & 0.5 & 0.6 \\ 0.4 & 0.5 & 0.6 & 0.7 \\ 0.501 & 0.6 & 0.7 & 0.8 \end{bmatrix}$$

1) Calculate the inverse A^{-1} of A.

2) Calculate the following condition number: $cond(A)_1$.

   Is A well- or ill-conditioned?

PROBLEM No. 23 / I

Consider the HILBERT matrix of order 5:

$$H_5 = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix};$$

1) Solve the linear system $H_5 x = b$, for:

   $b = [1.0 \quad 0.6 \quad 0.4 \quad 0.3 \quad 0.3]^T$, and $\tilde{b} = [1.02 \quad 0.6 \quad 0.4 \quad 0.3 \quad 0.3]^T$.

2) Compute the ratio: maximum relative perturbation in solution (in modulus) / relative perturbation in right-hand side b_1 (in modulus). Comment the result. (If a perturbed becomes $\tilde{a}$, the relative perturbation is: $(\tilde{a} - a)/a$.)

PROBLEM No. 24 / I

Consider the linear system $Ax = b$, where:

$$A = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix}, \text{ and } b = \left[\begin{array}{c:c} 1 & 1 \\ -2 & 0 \\ -2 & 1 \\ -2 & 0 \\ 1 & 1 \end{array}\right].$$

1) Solve the system by LU method.
2) How can be computed the determinant of A?

PROBLEM No. 25 / I

Consider the HILBERT matrix of order 3:

$$H_3 = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix};$$

Matrix entries are: $h_{ij} = 1./(i + j - 1)$.

1) Calculate H_3^{-1} in single precision.

2) Calculate H_3^{-1} in double precision.

3) Calculate in single precision the *analytical* inverse $\left(H_3^{-1}\right)_T = [\alpha_{ij}]$, where:

$$\alpha_{ij} = (-1)^{i+j} \frac{(n+i-1)!(n+j-1)!}{(i+j-1)[(i-1)!(j-1)!]^2 (n-i)!(n-j)!}$$

Compare the elements of the inverses in part (1) and (2), with those of the inverse in part (3). Comment the comparison results.

PROBLEM No. 26 / I

Consider the following linear system $Ax = b$:

$$A = \begin{bmatrix} 6 & 1 & 1 & 0 & 0 \\ 1 & 6 & 1 & 1 & 0 \\ 1 & 1 & 6 & 1 & 1 \\ 0 & 1 & 1 & 6 & 1 \\ 0 & 0 & 1 & 1 & 6 \end{bmatrix}; \qquad b = \left[\begin{array}{c:c} 1 & 0.8 \\ 1 & 0.9 \\ 1 & 1. \\ 1 & 0.9 \\ 1 & 0.8 \end{array}\right];$$

Matrix A is positive definite.

1) Solve the system by CHOLESKY Method.

2) Print the matrix L, and calculate the determinant of A.

PROBLEM No. 27 / I

Given the LOTKIN matrix of order 5:

$$A_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1./2 & 1./3 & 1./4 & 1./5 & 1./6 \\ 1./3 & 1./4 & 1./5 & 1./6 & 1./7 \\ 1./4 & 1./5 & 1./6 & 1./7 & 1./8 \\ 1./5 & 1./6 & 1./7 & 1./8 & 1./9 \end{bmatrix};$$

(Matrix entries are: $a_{1j} = 1;\quad a_{ij} = 1./(i + j - 1) \ldots i > 1;\quad j = \overline{1,5}$.)

1) Compute A_5^{-1} in single precision, inputting A elements in rows 2...5, in two modes:
   a) Element values, rounded to 7 significant digits;
   b) The inverse of the denominators, computed by the program (via the code).
2) Compare the elements of the inverse matrices in part a) and b). Explain why differences occur.

Note: The elements of the inverse are known analytically, and are integers ■

PROBLEM No. 28 / I

Generate the matrix A of order 5, with the following entries (Cauchy):

$a_{ij} = 1./(\, x(i) + y(j)\,),\quad i, j = \overline{1,5}$

where: $x(i) = 2i - 1;\quad y(j) = 2j$.

Working in single precision, complete the following tasks:

1) Print matrix A;
2) Calculate the inverse A^{-1};
3) Is the matrix A well- or ill-conditioned?

PROBLEM No. 29 / I

Given the matrix:

$$A = \begin{bmatrix} 3.00 & -1.05 & 2.53 \\ 4.33 & 0.560 & -1.78 \\ -0.830 & -0.540 & 1.47 \end{bmatrix}$$

- Calculate the condition number $cond(A)_*$.
- Is A well- or ill-conditioned?

PROBLEM No. 30 / I

Given the 4×4 matrix A, with the following entries:

0.715	5.280	3.795	0.210
0.495	4.840	4.335	0.330
0.330	4.335	4.840	0.495
0.210	3.795	5.280	0.715

- Calculate the condition number $cond(A)_1$.
- Is A well- or ill-conditioned?