PROBLEM No. 1 / II

Given a real number $x > 0$, let $y(x)$ be the smallest (positive) number, that added to *x* gives a result which does not round to *x* (is greater than *x*).

Write a program which determine $y(x)$ for $x = 2, 3, \ldots, 18$ (step 1), and verify that $x + y(x) > x$.

PROBLEM No. 2 / II

Calculate the values of the following functions, at values $x = 10^i, \quad i = 1, 2, \ldots, 7$.

$f(x) = x(\sqrt{x^2 + 4} - \sqrt{x^2 - 3})$ - In single precision;

$f2(x) = x(\sqrt{x^2 + 4} - \sqrt{x^2 - 3})$ - In double precision;

$g(x) = \dfrac{7x}{\sqrt{x^2 + 1} + \sqrt{x^2 - 3}}$ - In single precision.

- Tabulate computed values ($f_2(x)$ with 15 significant digits), and explain the results.
- Find out the number of correct significant digits of $f(x)$ value for $x = 10^3$.

PROBLEM No. 3 / II

Equation $f(x) = e^x - x^4 = 0$ has two roots.

- Find intervals of length 1, containing the roots.
- Find the roots by BISECTION method, with tolerance $EPS = 10^{-6}$.
- Find again the roots, by SECANT method, with the same tolerance, and compare the number of iterations.

PROBLEM No. 4 / II

Given the equation $f(x) = 0$, where:

$f(x) = 0.9\cos(x) - x + 1.6$

1) Solve the equation by NEWTON method, taking $x_0 = 1.5$ and tolerance $EPS = 10^{-6}$.
2) Solve the equation by SECANT method, with $x_0 = 1.5$, $x_1 = 1.6$, and $EPS = 10^{-6}$.
3) Compare the number of iterations in the two methods.

PROBLEM No. 5 / II

Given the equation:

$$tg(x) = \frac{1.82 - x}{x - 0.2}$$

Solve the equation for a root near $x_0 = 0.8$ with tolerance $EPS = 10^{-6}$, by:

1) NEWTON method;
2) SECANT method;
3) Compare the number of iterations in the two methods.

PROBLEM No. 6 / II

Consider the function

$$f(x) = -x + e^{-px^3}\cos(x),$$

where p is a parameter.

The equation $f(x) = 0$ has a unique root in the interval $(0, 1)$.

1) Find the roots, with a tolerance $EPS = 10^{-6}$, for the following p values: p = 1; 5; and 25.
2) Solve by Newton method the case $p = 25$, with $EPS = 10^{-6}$ and $x_0 = 0$.
3) Comment the result.

PROBLEM No. 7 / II

Given the annuity equation

$$P_1[(1+r)^{N_1}-1]=P_2[1-(1+r)^{-N_2}];$$

Find r for values: $N_1=35$, $N_2=25$, $P_1=5000$, and $P_2=10000$, by:

1) Newton method;

2) Secant method.

(Meaning: r = yearly nominal interest rate; P_1 = amount of deposit at the beginning of years $1, 2, \ldots, N_1$; P_2 = amount of the payment at the beginning of years $N_1+1, N_2+1, \ldots, N_1+N_2$. After the last payment, the account balance is zero.)

PROBLEM No. 8 / II

Consider the equation $x=g(x)$, where

$$g(x)=\frac{1.15-0.69tg(x)}{1+tg(x)}$$

1) Iterate in single precision with $x_0=0.6$, tolerance $EPS=10^{-6}$, and limited number of iterations $NLIM\geq 2000$.

2) Repeat the iteration in double precision, and find the root with tolerance $EPS=10^{-9}$ (Allow a sufficient number of iterations!).

3) Explain why a very large number of iteration is needed, before meeting a stopping criterion of the iteration.

PROBLEM No. 9 / II

Given the CHEBISHEV polynomial of 2nd kind, of order 6:

$T_6(x) = 64x^6 - 80x^4 + 24x^2 - 1.$

1) Find initial approximations of the roots (by algebraic methods, function graph, etc.)
2) Calculate the roots, with tolerance 10^{-6}.
3) *Note*: All roots are of modulus < 1.

PROBLEM No. 10 / II

The polynomial

$p(x) = x^5 - 18x^4 + 118x^3 - 348x^2 + 457x - 210$

Has the roots: $x_1 = 1;\ x_2 = 2;\ x_3 = 3;\ x_4 = 5;\ x_5 = 7$.

Let $\tilde{p}(x)$ be the polynomial obtained form $p(x)$ by replacing the coefficient $a_3 = 118$ of x^3, with $\tilde{a}_3 = 118.02$.

- Calculate the roots of $\tilde{p}(x)$.
- Calculate the ratio: maximum relative perturbation in the roots (in modulus) / the relative perturbation (in modulus) in coefficient a_3. Comment the result. (If *a* perturbed becomes $\tilde{a}$, the relative perturbation is: $(\tilde{a} - a)/a$.)

PROBLEM No. 11 / II

In the problem of missile interception, the following system is obtained:

$$\begin{cases} t\cos(\alpha) + t - 1 = 0 \\ t\sin(\alpha) - 0.1t^2 + e^{-t} - 1 = 0 \end{cases}$$

(t is the time, and α is the firing angle of the interceptor.)

Solve the system, with tolerance $EPS = 10^{-6}$ and initial approximation

$w_0 = (0.5, 1)$, by:

1) Iteration with constant matrix A (with updating after 3 steps)

2) NEWTON method.

PROBLEM No. 12 / II

Consider the system:

$$\begin{cases} \sin(x+y) = x + 0.1 \\ \cos(x-y) = y + 0.5 \end{cases}$$

- Solve the system by NEWTON method, with tolerance $EPS = 10^{-6}$: Find the root near $w_0 = (1, 0)$

- Solve again, by iteration with constant matrix A (with updating after 3 steps).

PROBLEM No. 13 / II

Consider the system:

$$\begin{cases} f_1(x,y) = x^2 + 4y^2 - 16 = 0 \\ f_2(x,y) = -x^2 + y + 3 = 0 \end{cases}$$

Solve by NEWTON method, with tolerance $EPS = 10^{-6}$. Find the initial approximations x_0, y_0, from the intersection of the curves $f_1(x,y) = 0$ and $f_2(x,y) = 0$.

PROBLEM No. 14 / II

Consider the system:

$$\begin{cases} x + y + z^2 = 3.8 \\ xyz = -1.9 \\ \sqrt{x} + \sqrt{y} - z^2 = 1.3 \end{cases}$$

Find the root near $w_0 = (1.5, 1, -1)$, with a tolerance $EPS = 10^{-6}$.

Use either iteration with constant matrix A (updated after 3 steps), or Newton method.

PROBLEM No. 15 / II

Consider the system:

$$\begin{cases} (x-y)(x+y)^{1/2} = 3 \\ x - \log(x-y) = 1 \end{cases}$$

Solve the system, with:

- Tolerance $EPS = 10^{-6}$.
- Initial approximation $w_0 = (2, -0.5)$.

Use either iteration with constant matrix A (updated after 3 steps), or Newton method.

PROBLEM No. 16 / II

Consider the system:

$$\begin{cases} x^2 + 2\sin(y) + z = -0.1 \\ \cos(y) - z = 2.1 \\ x^2 + y^2 + z^2 = 2 \end{cases}$$

Find the solution near $w_0 = (1, 0, -1)$, with tolerance $EPS = 10^{-6}$, either by Newton method, or by iteration with constant matrix A (with updating after 3 steps).

PROBLEM No. 17 / II

Consider the linear system $Ax = b$, where:

$$A = \begin{bmatrix} 16 & 4 & 8 \\ 4 & 5 & -4 \\ 8 & -4 & 22 \end{bmatrix}, \text{ and } b = \left[\begin{array}{c:c:c} 1 & 2 & 28 \\ 1 & 1 & 5 \\ 1 & -1 & 26 \end{array}\right].$$

1) Solve the system by LU method.

2) Compute the determinant of A.

PROBLEM No. 18 / II

Given the linear system $Ax = b$, where:

$$A = \begin{bmatrix} 6 & -36 & 12 \\ -36 & 218 & -74 \\ 12 & -74 & 64 \end{bmatrix}; \qquad b = \begin{bmatrix} 1 & -1.8 \\ 1 & 10.8 \\ 1 & 0.2 \end{bmatrix};$$

Matrix A is positive definite.

3) Solve the system by CHOLESKY Method.

4) Print the matrix L, and calculate the determinant of matrix A.

PROBLEM No. 19 / II

Consider the linear system $Ax = b$, where:

$$A = \begin{bmatrix} 2 & 3 & 4 & 5.01 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5.01 & 6 & 7 & 8 \end{bmatrix}; \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

1) Calculate the solution for the right-hand sides $b = [1 \quad 2 \quad 3 \quad 4]^T$, and

   $\tilde{b} = [1 \quad 2 \quad 3 \quad 4.01]^T$

2) Calculate the ratio: maximum relative perturbation (in modulus) in solution / relative perturbation in the right-hand side b_4. Comment the result.

   (If a perturbed becomes $\tilde{a}$, the relative perturbation is: $(\tilde{a} - a)/a$.)

PROBLEM No. 20 / II

Given the matrix:

$$A = \begin{bmatrix} 2 & 3 & 4 & 5.01 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5.01 & 6 & 7 & 8 \end{bmatrix}$$

1) Calculate the inverse A^{-1} of A.

2) Calculate the condition number: $cond(A)_\infty$.

PROBLEM No. 21 / II

Given the matrix

$$A = \begin{bmatrix} 5 & 6 & 7 & 8.01 \\ 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 10 \\ 8.01 & 9 & 10 & 11 \end{bmatrix}$$

1) Calculate the inverse A^{-1} of A.

2) Calculate the condition number: $cond(A)_1$.

PROBLEM No. 22 / II

Given the HILBERT matrix of order 4:

$$H_4 = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$$

1) Solve the linear system $H_4 x = b$, for:

   $b = [1 \quad 0.2 \quad 0.3 \quad 0.4]^T$, and $b = [1.02 \quad 0.2 \quad 0.3 \quad 0.4]^T$.

2) Calculate the ratio: the maximum relative perturbation (in modulus) in solution / the relative perturbation in the right-hand side b_1 (in modulus). Comment the result.

   (If a perturbed becomes $\tilde{a}$, the relative perturbation is: $(\tilde{a} - a)/a$.)

PROBLEM No. 23 / II

Given the following matrix of order 5:

$$A_5 = \begin{bmatrix} 1./2 & 1./2 & 1./2 & 1./2 & 1./2 \\ 1./2 & 1./3 & 1./4 & 1./5 & 1./6 \\ 1./3 & 1./4 & 1./5 & 1./6 & 1./7 \\ 1./4 & 1./5 & 1./6 & 1./7 & 1./8 \\ 1./5 & 1./6 & 1./7 & 1./8 & 1./9 \end{bmatrix};$$

(Matrix elements are: $a_{1j} = 1./2; \quad a_{ij} = 1./(i + j - 1) \ldots i > 1; \quad j = \overline{1,5}$.)

1) Compute A_5^{-1} in single precision, inputting A elements in rows 2...5, in two modes:
   a) Computed values, with 7 correct significant digits;
   b) The inverse of the denominators, computed by the program (via the code).
2) Explain why differences occur, between the elements of the inverses in part a) and b).

PROBLEM No. 24 / II

Given the following matrix of order 5:

$$A_5 = \begin{bmatrix} 1./2 & 1./2 & 1./2 & 1./2 & 1./2 \\ 1./2 & 1./3 & 1./4 & 1./5 & 1./6 \\ 1./3 & 1./4 & 1./5 & 1./6 & 1./7 \\ 1./4 & 1./5 & 1./6 & 1./7 & 1./8 \\ 1./5 & 1./6 & 1./7 & 1./8 & 1./9 \end{bmatrix};$$

(Matrix elements are: $a_{1j} = 1./2; \quad a_{ij} = 1./(i + j - 1) \ldots i > 1; \quad j = \overline{1,5}$.)

1) Compute the condition numbers $cond(A_5)_\infty$ and $cond(A_5)_1$, in single precision, working with A_5 elements computed in two modes:
   a) Element values, rounded to 7 significant digits;
   b) Inverting the denominators by the program (via the code).
2) Compare the results in part a) and b). Explain why they are different.

PROBLEM No. 25 / II

Given the following matrix of order 5 (Moler2):

$$A_5 = \begin{bmatrix} -9.00000 & 11.00000 & -21.00000 & 63.00000 & -252.0000 \\ 70.0000 & -69.00000 & 141.0000 & -421.0000 & 1684.000 \\ -575.000 & 575.000 & -1149.000 & 3451.000 & -13801.00 \\ 3891.000 & -3891.000 & 7782.000 & -23345.000 & 93365.00 \\ 1024.000 & -1024.000 & 2048.000 & -6144.000 & 24572.00 \end{bmatrix}$$

1) Solve the system $A_5 x = b$, choosing the RHS b at will.

2) Let $\tilde{A}_5$ be the matrix obtained from A_5, replacing element $a_{22} = -69.00$, with $\tilde{a}_{22} = -69.02$ (All other elements remain unchanged). Solve the system $\tilde{A}_5 x = b$.

1) Compare the solutions. Is the matrix A_5 well- or ill-conditioned?

3) <u>Note:</u> Modify the threshold (datum `prag)` in `Main-Elim` (or `LU-Decomp`) to `1E-8` !

PROBLEM No. 26 / II

Given the following matrix of order 5 (Moler):

$$A_5 = \begin{bmatrix} 1.000000 & 3.000000 & 3.000000 & 3.000000 & 3.000000 \\ 3.000000 & 10.000000 & 12.000000 & 12.000000 & 12.000000 \\ 3.000000 & 12.000000 & 19.000000 & 21.000000 & 21.000000 \\ 3.000000 & 12.000000 & 21.000000 & 28.000000 & 30.000000 \\ 3.000000 & 12.000000 & 21.000000 & 30.000000 & 37.000000 \end{bmatrix}$$

1) Calculate the inverse A_5^{-1}.

2) Is the matrix A_5 well- or ill-conditioned?

3) Compute also, the determinant of the matrix.

<u>Note:</u> Modify the threshold (datum `prag)` in `Main-Elim` (or `LU-Decomp`) to `1E-8` !

PROBLEM No. 27 / II

Given the CAUCHY matrix of order 5:

$$A_5 = \begin{bmatrix} 0.333333 & 0.200000 & 0.142857 & 0.090909 & 0.100000 \\ 0.200000 & 0.142857 & 0.111111 & 0.076923 & 0.083333 \\ 0.142857 & 0.111111 & 0.090909 & 0.066667 & 0.071429 \\ 0.100000 & 0.083333 & 0.071429 & 0.055556 & 0.058824 \\ 0.111111 & 0.090909 & 0.076923 & 0.058824 & 0.062500 \end{bmatrix}$$

1) Calculate the inverse A_5^{-1}.

2) Is the matrix A_5 well- or ill-conditioned?

3) Compute also, the determinant of the matrix.

Note: Modify the threshold (datum `prag)` in `Main-Elim` (or `LU-Decomp`) to `1E-8` !

PROBLEM No. 28 / II

Given the matrix *A* of order 5 with the entries (Wilkinson 5 - modified):

$$a_{ij} = 3.6288/(i + j + 1), \quad i, j = \overline{1,5}$$

Working in single precision, it is required:

1) Print matrix *A*;

2) Calculate the inverse A^{-1};

3) Is the matrix well- or ill-conditioned?

Note: Modify the threshold (datum `prag)` in `Main-Elim` (or `LU-Decomp`) to `1E-8` !

PROBLEM No. 29 / II

Given the matrix:

$$A = \begin{bmatrix} 0.205 & 0.310 & 0.410 \\ 0.310 & 0.405 & 0.510 \\ 0.410 & 0.510 & 0.605 \end{bmatrix}$$

- Calculate the condition number $cond(A)_*$.
- Is A well- or ill-conditioned?

PROBLEM No. 30 / II

Consider the following matrix:

$$A = \begin{bmatrix} 5 & 6 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ 2 & 2 & 1 & 6 \\ 4 & 2 & 3 & 4.02 \end{bmatrix}$$

- Calculate the condition number $cond(A)_1$.
- Is A well- or ill-conditioned?