PROBLEM No. 1/ I

The following numerical data are represented, in the <u>single format</u>:

$x = 1E37$; $x = 5E38$;

$x = 5E-37$; $x = 1E-39$; $x = 1E-45$;

1) Find the following representation parameters:

   - Sign; Exponent field; Significand field;
   - Biased (Stored) exponent; Representation Exponent.
   - Significand (*sig*), and fraction in the model (*fr*). For a finite numerical data we have $sig = 2 * fr$. Why ?

2) Specify the class of each datum *x*.

PROBLEM No. 2 / I

For the following numbers:

$x = 1.0$; $x = 3.0 \times 10^{16}$

1) Calculate: $ULP(x)$, $x_1 = x + ULP(x)$, and $x_2 = x + \frac{1}{2}ULP(x)$.

   Check if $x_2 = x$ or $x_2 > x$.

2) Find the floating-point numbers <u>nearest</u> to *x*, namely:

   x^- = nearest and smaller than *x*;

   x^+ = nearest and larger than *x*.

   Calculate the differences: $x - x^-$ and $x^+ - x$, and express them as a multiple (or fraction) of $ULP(x)$.

PROBLEM No. 3 / I

Calculate the values of functions *f*, *f2*, and *g*, at the following *x* values:

$x = 10^i, \quad i = 1, 2, \ldots, 7$.

1) $f(x) = x^2(\sqrt{x+10} - \sqrt{x-5})$ ... In single precision;

2) $f2(x) = x^2(\sqrt{x+10} - \sqrt{x-5})$ … In the highest available precision;

3) $g(x) = \dfrac{15x^2}{\sqrt{x+10} + \sqrt{x-5}}$ … In single precision.

- Tabulate computed values: *f* and *g* – with 7 significant digits; $f2$ with 8 significant digits; explain the differences between values in (1) and (2).
- Find out the number of correct significant digits of $f(x)$ value, for $x = 10^6$.

PROBLEM No. 4 / I

The equation $f(x) = e^x - 4x^2 = 0$ has three roots.

Let x_0 be an approximation of the root.

Find the three roots, with tolerance $EPS = 10^{-6}$, by:

1) SECANT Method: with initial approximations $x_0 \pm h$, where $0 < h \le x_0 / 10$;

2) NEWTON Method: initial approximation x_0;

3) A 3rd order method: initial approximation x_0.

Compare the number of iterations, needed to meet tolerance *EPS*.

PROBLEM No. 5 / I

Consider the equation $f(x) = 0$, where:

$$f(x) = x + x^2 - (\sin(x))^2 - 1$$

Find two initial approximations x_0.

1) Solve the equation by NEWTON method, choosing the minimum tolerance *EPS*.
2) Solve again, for the negative root, by the fixed-point method. Use the explicit procedure $g(x) = x - mf(x)$, namely:
   - Choose an *m* in the range 0.01…0.1; the iteration does not converge to the negative root. Why?
   - Choose *m* such that the iteration converge the most rapidly. Use the minimum tolerance *XTOL*. Compare the number of iterations with this number in Newton method (1).

PROBLEM No. 6 / I

Given the equation

$$tg(x) = \frac{8 - x}{x + 0.2}.$$

1) Solve the equation for the root near $x_0 = 4$, by NEWTON method. Establish and use, the minimum tolerance $EPS_{\min}$.
2) Solve the equation by the fixed-point method; use the minimum tolerance $XTOL_{\min}$. Compare the number of iterations in the two methods.

PROBLEM No. 7 / I

Consider the function

$f(x) = x + e^{-px^2} \cos(x)$,

where p is a parameter.

Equation $f(x) = 0$ has a unique root in the interval $(-1, 0)$.

1) Find the roots for the following p values: $p = 1; 5;$ and 25, with a tolerance

   $EPS = 10^{-6}$.

2) Solve by Newton method the case $p = 25$, with $EPS = 10^{-6}$, and $x_0 = 0$.

   Comment the result.

PROBLEM No. 8 / I

The equation $e^x - 5x^2 = 0$ has a root in the interval $[4.5, 5]$.

Consider the equation put in the form $x = g(x)$, where:

1) $g(x) = \sqrt{e^x / 5}$

Iterate (in the fixed-point method) with $x_0 = 4.7$. The iteration does not converge.

Why?

2) $g(x) = x - m(e^x - 5x^2)$

   - Determinate *m* such that the fixed-point iteration converge, and compute the root.
   - Find the value of *m*, for which the iteration converges the most rapidly.

PROBLEM No. 9 / I

Given the equation $x = g(x)$, where

$$g(x) = \frac{2.1 + 0.2tg(x)}{1 + tg(x)};$$

1) Iterate in single precision with $x_0 = 1.$, tolerance $EPS = 10^{-6}$, and limited number of iterations $NLIM \geq 2000$.
2) Repeat the iteration in double precision and find the root with tolerance $EPS = 10^{-9}$.
3) Explain why a large number of iterations is needed, before meeting the iteration halting criterion.

PROBLEM No. 10 / I

Determine *t*, with a tolerance of 10^{-5}, from the equation

$$e^{-t/2} \cosh^{-1}(e^{t/2}) = \sqrt{0.5L_{cr}},$$

for $L_{cr} = 0.088$. (*t* is the temperature in the interior of a material with imbedded heat sources; Frank-Kamenetski, 1955).

Use two numerical methods (chosen at will), with: same tolerance; same or close initial approximation(s). Compare the results.

PROBLEM No. 11 / I

Given the LAGUERRE polynomial of order 5:

$$L_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)$$

1) Find initial approximations of the roots (by algebraic methods, function graph, etc.)
2) Calculate the roots, with tolerance 10^{-6}.
3) Study the stability of the roots.

PROBLEM No. 12 / I

Given the LEGENDRE polynomial of order 6

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5),$$

1) Find initial approximations of the roots (by algebraic methods, function graph, etc.)
2) Calculate the roots, with tolerance 10^{-6}.

Hint: All roots are of modulus < 1.

3) Study the stability of the roots.

PROBLEM No. 13 / I

The polynomial

$$p(x) = x^6 - 21x^5 + 175x^4 - 735x^3 + 1624x^2 - 1764x + 720$$

has the roots: $x_i = i, i = \overline{1,6}$.

Let $\tilde{p}(x)$ be the polynomial obtained form $p(x)$ by replacing the coefficient

$a_5 = -21$ of x^5, with $\tilde{a}_5 = -21.002$. Let $\varepsilon = \tilde{a}_5 - a_5$ be the perturbation in a_5.

1) Calculate the roots $z_1(\varepsilon)$ of $\tilde{p}(x)$;

2) For the last two roots of $p(x)$, $z_1 = 5; 6$: Check the evaluation

   $z_1(\varepsilon) - z_1 \approx K^{(1)}(z_1)\varepsilon$, where $K^{(1)}(z) = -q(z)/p'(z)$;

3) Calculate the ratio: largest relative perturbation (in modulus) in the roots / relative perturbation (in modulus) in coefficient a_5. Comment the result.

PROBLEM No. 14 / II

The polynomial

$$p(x) = x^5 - 18x^4 + 118x^3 - 348x^2 + 457x - 210$$

has the roots: $x_1 = 1; x_2 = 2; x_3 = 3; x_4 = 5; x_5 = 7$.

Let $\tilde{p}(x)$ be the polynomial obtained form $p(x)$ by replacing the coefficient

$a_3 = 118$ of x^3, with $\tilde{a}_3 = 118.02$. Denote by $\varepsilon = \tilde{a}_3 - a_3$ the coefficient perturbation.

1) Calculate the roots $z_1(\varepsilon)$ of $\tilde{p}(x)$.

2) For the last two roots z_1 of $p(x)$: Check the evaluation

   $z_1(\varepsilon) - z_1 \approx K^{(1)}(z_1)\varepsilon$, where $K^{(1)}(z) = -q(z)/p'(z)$;

3) Calculate the ratio: maximum relative perturbation in the roots (in modulus) / the relative perturbation (in modulus) in coefficient a_3. Comment the result.

PROBLEM No. 15 / I

Consider the system:

$$\begin{cases} xy - z^2 = 2 \\ -xyz - x^2 + y^2 = 4 \\ e^x - e^y - z = 7 \end{cases}$$

Solve with tolerance $EPS = 10^{-6}$, by any method for systems (provided that the solution is obtained):

1) With initial approximation $w_0 = (1, 1, 1)$

2) With initial approximation $w_0 = (2, 2, -1)$

Compare the number of iterations and explain.

PROBLEM No. 16 / I

Consider the system:

$$\begin{cases} x - y + \sqrt{x+y} = 1 \\ x + y - e^{x-y} = 3 \end{cases}$$

- Find an initial approximation $w_0 = (x_0, y_0)$;
- Solve the system, with tolerance $EPS = 10^{-6}$, by:
  - Iteration with constant matrix A (updated after 3 steps);
  - Newton method (with analytical jacobian)

  Compare the number of iteration and comment.

PROBLEM No. 17 / I

Consider the system:

$$\begin{cases} x^2 + x - y^2 = 1 \\ y - \sin(x)^2 = 0 \end{cases}$$

Find two sets of initial approximations, and the solution in their neighborhood.

Choose a tolerance $EPS = 10^{-6}$, and solve by:

1) NEWTON method;
2) Iteration with constant matrix *A* (with updating after 3 steps).

Compare the number of iterations and explain.

PROBLEM No. 18 / I

Consider the system:

$$\begin{cases} 3x_1 - \cos(x_2 x_3) = 0.5 \\ x_1{}^2 - 81x_2{}^2 + \sin(x_3) = -1 \\ e^{-x_1 x_2} + 20x_3 = -9.5 \end{cases}$$

1) Find an initial approximation $\mathbf{x}^0 = (x_1^0, x_2^0, x_3^0)$, knowing that x_2 has a small value;
2) Solve by NEWTON method, with tolerance $EPS = 10^{-6}$.

PROBLEM No. 19 / I

Consider the system:

$$\begin{cases} \sin(xy) = 0.5 \\ \cos(x) = e^{y} \end{cases}$$

1) Find two initial approximations (x^0, y^0), in the domain $x \in [-1, 5]$, $y \in [-1.5, -0.5]$;

2) Find the solutions near (x^0, y^0), by:

   - NEWTON method, with tolerance $EPS = 10^{-6}$;
   - Iteration with constant matrix A (updated after 3 steps).

   Compare the number of iterations and explain.

PROBLEM No. 20 / I

Consider the system:

$$\sin(\pi xy) - 0.5y - x = 0$$
$$(1 - 0.25/\pi)(e^{2x-1} - 1) + y - 2x = 0$$

With tolerance $EPS = 10^{-6}$, and for the domain $x \in [0.25, 1]$, and $y \in [0.5, 2]$:

Find the two solutions of the system.

PROBLEM No. 21 / I

Given the linear system with matrix:

$$\mathbf{A} = \begin{bmatrix} 3.01 & 6.03 & 1.99 \\ 1.27 & 4.16 & -1.23 \\ .987 & -4.81 & 9.34 \end{bmatrix}$$

1) Calculate the solution of $\mathbf{Ax} = \mathbf{b}$ for the right-hand side $\mathbf{b} = [1 \quad 1 \quad 1]^T$.

2) Repeat part (1), with matrix $\mathbf{A}'$ obtained from **A**, by the replacements: $3.01 \to 3.00$ (element a_{11}) and $.987 \to .990$ (element a_{31}). Compare the results and conclude about the conditioning of the problem.

PROBLEM No. 22 / I

Given the matrix:

$$A = \begin{bmatrix} 3.01 & 6.03 & 1.99 \\ 1.27 & 4.16 & -1.23 \\ .987 & -4.81 & 9.34 \end{bmatrix}.$$

1) Calculate the inverse A^{-1} of A.

2) Calculate the condition number: $cond(A)_\infty$. Is A well- or ill-conditioned?

PROBLEM No. 23 / I

Given the matrix

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.501 \\ 0.3 & 0.4 & 0.5 & 0.6 \\ 0.4 & 0.5 & 0.6 & 0.7 \\ 0.501 & 0.6 & 0.7 & 0.8 \end{bmatrix}$$

1) Calculate the inverse A^{-1} of A.

2) Calculate the following condition number: $cond(A)_1$.

   Is A well- or ill-conditioned?

PROBLEM No. 24 / I

Consider the HILBERT matrix of order 5:

$$H_5 = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix};$$

1) Solve the linear system $H_5 x = b$, for:

   $b = [1.0 \quad 0.6 \quad 0.4 \quad 0.3 \quad 0.3]^T$, and $\tilde{b} = [1.02 \quad 0.6 \quad 0.4 \quad 0.3 \quad 0.3]^T$.

2) Compute the ratio: maximum relative perturbation in solution (in modulus) / relative perturbation in right-hand side b_1 (in modulus). Comment the result.

PROBLEM No. 25 / I

Consider the linear system $Ax = b$, where:

$$A = \begin{bmatrix} -1 & 4 & -1 & 0 & 0 \\ 4 & -1 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix}, \text{ and } b = \left[\begin{array}{c:c} 1 & 1 \\ -2 & 0 \\ -2 & 1 \\ -2 & 0 \\ 1 & 1 \end{array}\right].$$

1) Solve the system by LU method: use partial pivoting, with the maximum pivot.
2) How can be computed the determinant of A?
3) Print the product $L*U$. Which is the relation of $L*U$ with the matrix A?

PROBLEM No. 26 / I

Consider the HILBERT matrix of order 3:

$$H_3 = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix};$$

Matrix entries are: $h_{ij} = 1./(i + j - 1)$.

1) Calculate H_3^{-1} in single precision.
2) Calculate H_3^{-1} in double precision.
3) Calculate in single precision the *analytical* inverse $\left(H_3^{-1}\right)_T = [\alpha_{ij}]$, where:

$$\alpha_{ij} = (-1)^{i+j} \frac{(n+i-1)!(n+j-1)!}{(i+j-1)[(i-1)!(j-1)!]^2 (n-i)!(n-j)!}$$

Compare the elements of the inverses in part (1) and (2), with those of the inverse in part (3). Comment the comparison results.

PROBLEM No. 27 / I

Consider the following linear system $Ax = b$:

$$A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 4 & 1 & -1 \\ -1 & 1 & 4 & 1 \\ -1 & -1 & 1 & 4 \end{bmatrix}; \qquad b = \begin{bmatrix} 0.8 \\ 0.9 \\ 1 \\ 0.8 \end{bmatrix};$$

1) Calculate all principal minors of A – that is the determinants of sub-matrices

   $A(1:k,1:k), \quad k = \overline{1,4}$: For $k \geq 3$ use a program which calculates the determinant.

2) Replace elements $a_{12} = a_{21} = 1$ by $a_{12} = a_{21} = 4.5$, and denote by A' the new matrix. Try to solve $Ax = b$: the Cholesky decomposition of A' is not feasible. Why?

PROBLEM No. 28 / I

Consider the following linear system $Ax = b$:

$$A = \begin{bmatrix} 5 & 2 & -2 & 0 & 0 \\ 2 & 5 & 2 & -2 & 0 \\ -2 & 2 & 5 & 2 & -2 \\ 0 & -2 & 2 & 9 & 2 \\ 0 & 0 & -2 & 2 & 9 \end{bmatrix}; \qquad b = \left[\begin{array}{c:c} 1 & 0.8 \\ 1 & 0.9 \\ 1 & 1. \\ 1 & 0.9 \\ 1 & 0.8 \end{array}\right];$$

Matrix A is positive definite.

1) Solve the system by CHOLESKY Method.

2) Print the factor L (or S). How can be calculated the determinant of the matrix?

3) Change the value of $a_{44} = 9$ to $a_{44} = 6$, and denote the new matrix by A':

   (a) Try to solve the system $A'x = b$: it is not possible. Why?

   (b) Solve the system $A'x = b$ by another method.

PROBLEM No. 29 / I

Given the LOTKIN matrix of order 5:

$$A_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1./2 & 1./3 & 1./4 & 1./5 & 1./6 \\ 1./3 & 1./4 & 1./5 & 1./6 & 1./7 \\ 1./4 & 1./5 & 1./6 & 1./7 & 1./8 \\ 1./5 & 1./6 & 1./7 & 1./8 & 1./9 \end{bmatrix};$$

(Matrix entries are: $a_{1j} = 1; \quad a_{ij} = 1./(i + j - 1) \ldots i > 1; \quad j = \overline{1,5}$.)

1) Compute A_5^{-1} in single precision, inputting A elements in rows 2...5, in two modes:
   a) Element values, rounded to 7 significant digits;
   b) The inverse of the denominators, computed by the program (via the code).
2) Compare the elements of the inverse matrices in part a) and b). Explain why differences occur.

Note: The elements of the inverse are known analytically, and are integers ■

PROBLEM No. 30 / I

Generate the matrix A of order 5, with the following entries (Cauchy):

$a_{ij} = 1./(\ x(i) + y(j)\ ), \quad i, j = \overline{1,5}$

where: $x(i) = 2i - 1; \quad y(j) = 2j$.

Working in single precision, complete the following tasks:

1) Print matrix A;
2) Calculate the inverse A^{-1};
3) Is the matrix A well- or ill-conditioned?

PROBLEM No. 31 / I

Given the matrix:

$$A = \begin{bmatrix} 3.00 & -1.05 & 2.53 \\ 4.33 & 0.560 & -1.78 \\ -0.830 & -0.540 & 1.47 \end{bmatrix}$$

1) Calculate the condition number $cond(A)_*$.

2) Is A well- or ill-conditioned?

PROBLEM No. 32 / I

Given the 4×4 matrix A, with the following entries:

```
0.715        5.280        3.795        0.210
0.495        4.840        4.335        0.330
0.330        4.335        4.840        0.495
0.210        3.795        5.280        0.715
```

1) Calculate the condition number $cond(A)_1$.

2) Is A well- or ill-conditioned?

PROBLEM No. 33 / II

Given the following matrix of order 5 (Moler):

$$A_5 = \begin{bmatrix} 1.000000 & 3.000000 & 3.000000 & 3.000000 & 3.000000 \\ 3.000000 & 10.000000 & 12.000000 & 12.000000 & 12.000000 \\ 3.000000 & 12.000000 & 19.000000 & 21.000000 & 21.000000 \\ 3.000000 & 12.000000 & 21.000000 & 28.000000 & 30.000000 \\ 3.000000 & 12.000000 & 21.000000 & 30.000000 & 37.000000 \end{bmatrix}$$

1) Calculate the inverse A_5^{-1}.

2) Is the matrix A_5 well- or ill-conditioned?

3) Compute also, the determinant of the matrix.

<u>Note:</u> Modify the threshold (datum `prag`) in `Main-Elim` (or `LU-Decomp`) to `1E-8` !

PROBLEM No. 34 / II

Given the matrix *A* of order 5 with the entries (Wilkinson 5 - modified):

$$a_{ij} = 3.6288/(i + j + 1), \quad i, j = \overline{1,5}$$

Working in single precision, it is required:

1) Print matrix *A*;

2) Calculate the inverse A^{-1};

3) Is the matrix well- or ill-conditioned?

<u>Note:</u> Modify the threshold (datum `prag`) in `Main-Elim` (or `LU-Decomp`) to `1E-8` !

PROBLEM No. 35 / II

Given the following matrix of order 5 (Moler2):

$$A_5 = \begin{bmatrix} -9.00000 & 11.00000 & -21.00000 & 63.00000 & -252.0000 \\ 70.0000 & -69.00000 & 141.0000 & -421.0000 & 1684.000 \\ -575.000 & 575.000 & -1149.000 & 3451.000 & -13801.00 \\ 3891.000 & -3891.000 & 7782.000 & -23345.000 & 93365.00 \\ 1024.000 & -1024.000 & 2048.000 & -6144.000 & 24572.00 \end{bmatrix}$$

1) Solve the system $A_5 x = b$, choosing the RHS b at will.

2) Let $\tilde{A}_5$ be the matrix obtained from A_5, replacing element $a_{22} = -69.00$, with $\tilde{a}_{22} = -69.02$ (All other elements remain unchanged). Solve the system $\tilde{A}_5 x = b$.

3) Compare the solutions. Is the matrix A_5 well- or ill-conditioned?

<u>Note:</u> Modify the threshold (datum `prag`) in `Main-Elim` (or `LU-Decomp`) to `1E-8` !

PROBLEM No. 36 / I

Given the matrix:

$$A = \begin{bmatrix} 2.05 & 3.10 & 4.10 \\ 3.10 & 4.05 & 5.10 \\ 4.10 & 5.10 & 6.05 \end{bmatrix}$$

1) Calculate the condition number $cond(A)_*$.

2) Is A well- or ill-conditioned?

PROBLEM No. 37 / I

Given the 4×4 matrix:

$$A = \begin{bmatrix} 5 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 2 & -2 \end{bmatrix}$$

1) Calculate the inverse A^{-1};

2) Calculate the condition number $cond(A)_*$.