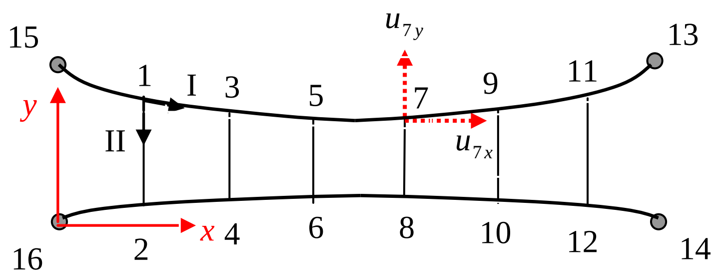
STRUCTURA MATRICII DE RIGIDITATE – Exemplu

Considerăm o fermă-cablu , cu numerotarea din figură a nodurilor.



Vectorul deplasărilor de nod va fi:

$$\mathbf{U}=\begin{bmatrix}\mathbf{U}_1\\ \cdots\\ \mathbf{U}_2\\ \cdots\\ \vdots\\ \cdots\\ \mathbf{U}_{12}\end{bmatrix}=\begin{bmatrix}u_{1x}\\ u_{1y}\\ \cdots\\ u_{2x}\\ u_{2y}\\ \cdots\\ \vdots\\ \cdots\\ u_{12x}\\ u_{12y}\end{bmatrix}=\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\\ \vdots\\ u_{23}\\ u_{24}\end{bmatrix}$$

Drumurile orientate, incidente intr-un nod, se definesc astfel:

I – În lungul cablului, sensul spre dreapta;

II – Vertical descendent.

Tabelul elementelor incidente în nod, pe drumurile I, II, este:

Nod	Drum incident în nod			
	I		II	
1	15-1	1-3	0	1-2
2	16-2	2-4	1-2	0
3	1-3	3-5	0	3-4
Etc.	...		...	

Notă

Elementele de cablu sunt notate acum, pentru o identificare mai simplă, prin *NI-NF* (nod_iniţial - nod_final); în practică, elementele se numerotează cu indici consecutivi 1 – Numărul_elementelor.

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Ecuaţiile de echilibru de nod sunt (6-2, §6.2, Teoria de calcul):

$$\left[-\mathbf{A}_{K-1}^{I} \;\vdots\; -\mathbf{A}_{K-1}^{II} \;\vdots\; \mathbf{A}_{K-1}^{I}+\mathbf{A}_{K-1}^{II}+\mathbf{A}_{K}^{I}+\mathbf{A}_{K}^{II} \;\vdots\; -\mathbf{A}_{K}^{I} \;\vdots\; -\mathbf{A}_{K}^{II}\right]\begin{bmatrix}\Delta\mathbf{U}_{K-1,I}\\ \Delta\mathbf{U}_{K-1,II}\\ \Delta\mathbf{U}_{K}\\ \Delta\mathbf{U}_{K+1,I}\\ \Delta\mathbf{U}_{K+1,II}\end{bmatrix}=\mathbf{P}_{K}-\mathbf{f}_{K}$$

Reamintim că, în aceste ecuaţii, indicele K notează:

- În $\mathbf{A}_{K}^{J}$: *elementul orientat* $K \rightarrow K+1$, pe drumul *J*.
- În $\Delta\mathbf{U}_{K}$: *nodul* K.

  În $\Delta\mathbf{U}_{K\pm1,J}$: indicii $K-1, J$ şi $K+1, J$ notează (relativ la nodul K): nodul "*în urmă*", şi respectiv nodul "*inainte*" , pe drumul J .

Ecuaţia anterioară se scrie, explicit:

$$\left[-\mathbf{A}_{K-1,K}^{I} \;\vdots\; -\mathbf{A}_{K-1,K}^{II} \;\vdots\; \mathbf{A}_{K-1,K}^{I}+\mathbf{A}_{K-1,K}^{II}+\mathbf{A}_{K,K+1}^{I}+\mathbf{A}_{K,K+1}^{II} \;\vdots\; -\mathbf{A}_{K,K+1}^{I} \;\vdots\; -\mathbf{A}_{K,K+!}^{II}\right]\begin{bmatrix}\Delta\mathbf{U}_{K-1,I}\\ \Delta\mathbf{U}_{K-1,II}\\ \Delta\mathbf{U}_{K}\\ \Delta\mathbf{U}_{K+1,I}\\ \Delta\mathbf{U}_{K+1,II}\end{bmatrix}=\mathbf{P}_{K}-\mathbf{f}_{K}$$

Astfel, avem:

- Nodul 1:

$$\left[-\mathbf{A}_{15-1} \;\vdots\; \mathbf{0} \;\vdots\; \mathbf{A}_{15-1}+\mathbf{0}+\mathbf{A}_{1-3}+\mathbf{A}_{1-2} \;\vdots\; -\mathbf{A}_{1-3} \;\vdots\; -\mathbf{A}_{1-2}\right]\begin{bmatrix}\Delta\mathbf{U}_{15}\\ \mathbf{0}\\ \Delta\mathbf{U}_{1}\\ \Delta\mathbf{U}_{3}\\ \Delta\mathbf{U}_{2}\end{bmatrix}=\mathbf{P}_1-\mathbf{f}_1$$

Ţineţi cont că $\mathbf{U}_{15}=\mathbf{0}$ (nodul 15 este fix), şi deci, $\Delta\mathbf{U}_{15}=\mathbf{0}$.

Ecuaţia efectivă, cu deplasările ordonate în ordinea $1-n$, va fi:

$$\left[-\mathbf{A}_{15-1} \;\vdots\; \mathbf{0} \;\vdots\; \mathbf{A}_{15-1}+\mathbf{0}+\mathbf{A}_{1-3}+\mathbf{A}_{1-2} \;\vdots\; -\mathbf{A}_{1-2} \;\vdots\; -\mathbf{A}_{1-3}\right]\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \Delta\mathbf{U}_{1}\\ \Delta\mathbf{U}_{2}\\ \Delta\mathbf{U}_{3}\end{bmatrix}=\mathbf{P}_1-\mathbf{f}_1$$

- Nodul 2:

$$\left[-\mathbf{A}_{16-2} \;\vdots\; -\mathbf{A}_{1-2} \;\vdots\; \mathbf{A}_{16-2}+\mathbf{0}+\mathbf{A}_{2-4}+\mathbf{A}_{1-2} \;\vdots\; -\mathbf{A}_{2-4} \;\vdots\; \mathbf{0}\right]\begin{bmatrix}\Delta\mathbf{U}_{16}\\ \Delta\mathbf{U}_{1}\\ \Delta\mathbf{U}_{2}\\ \Delta\mathbf{U}_{4}\\ \mathbf{0}\end{bmatrix}=\mathbf{P}_2-\mathbf{f}_2$$

- Nodul 3:

$$\left[-\mathbf{A}_{1-3} \;\vdots\; \mathbf{0} \;\vdots\; \mathbf{A}_{1-3}+\mathbf{0}+\mathbf{A}_{3-5}+\mathbf{A}_{3-4} \;\vdots\; -\mathbf{A}_{3-5} \;\vdots\; -\mathbf{A}_{3-4}\right]\begin{bmatrix}\Delta\mathbf{U}_{1}\\ \mathbf{0}\\ \Delta\mathbf{U}_{3}\\ \Delta\mathbf{U}_{5}\\ \Delta\mathbf{U}_{4}\end{bmatrix}=\mathbf{P}_3-\mathbf{f}_3$$

- Etc.

Ecuaţia matriceală de echilibru (ecuaţia 6-3 din § 6.2 – Teoria de calcul) este

$\mathbf{A}\cdot\Delta\mathbf{U}=\mathbf{P}-\mathbf{f}$

În care:

$$\Delta\mathbf{U}=\begin{bmatrix}\Delta\mathbf{U}_1\\ \hdashline \Delta\mathbf{U}_2\\ \hdashline \vdots\\ \hdashline \Delta\mathbf{U}_{12}\end{bmatrix}=\begin{bmatrix}\Delta u_{1x}\\ \Delta u_{1y}\\ \hdashline \Delta u_{2x}\\ \Delta u_{2y}\\ \hdashline \vdots\\ \hdashline \Delta u_{12x}\\ \Delta u_{12x}\end{bmatrix}=\begin{bmatrix}\Delta u_1\\ \Delta u_2\\ \Delta u_3\\ \Delta u_4\\ \vdots\\ \Delta u_{23}\\ \Delta u_{24}\end{bmatrix};\qquad \mathbf{P}-\mathbf{f}=\begin{bmatrix}\mathbf{P}_1-\mathbf{f}_1\\ \hdashline \mathbf{P}_2-\mathbf{f}_2\\ \hdashline \vdots\\ \hdashline \mathbf{P}_{12}-\mathbf{f}_{12}\end{bmatrix}.$$

Structura matricii de rigiditate tangente **A**, este:

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_{11} & -\mathbf{A}_{12} & -\mathbf{A}_{13} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ -\mathbf{A}_{21} & \mathbf{A}_{22} & -\mathbf{A}_{23} & -\mathbf{A}_{24} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ -\mathbf{A}_{31} & -\mathbf{A}_{32} & \mathbf{A}_{33} & -\mathbf{A}_{34} & -\mathbf{A}_{35} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathbf{A}_{42} & -\mathbf{A}_{43} & \mathbf{A}_{44} & -\mathbf{A}_{45} & -\mathbf{A}_{46} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ & & & & \ddots & & & & & & & \\ & & & & & & & & & \ddots & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{A}_{11,9} & -\mathbf{A}_{11,10} & \mathbf{A}_{11,11} & -\mathbf{A}_{11,12}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{A}_{12,10} & -\mathbf{A}_{12,11} & \mathbf{A}_{12,12}\end{bmatrix}$$

În care:

$\mathbf{A}_{11}=\mathbf{A}_{15-1}+\mathbf{A}_{1-3}+\mathbf{A}_{1-2};$

$\mathbf{A}_{12}=\mathbf{A}_{1-2};\quad \mathbf{A}_{13}=\mathbf{A}_{1-3}$

$\mathbf{A}_{22}=\mathbf{A}_{16-2}+\mathbf{A}_{2-4}+\mathbf{A}_{1-2}$

$\mathbf{A}_{21}=\mathbf{A}_{12};\quad \mathbf{A}_{23}=\mathbf{0};\quad \mathbf{A}_{24}=\mathbf{A}_{2-4}$

$\mathbf{A}_{33}=\mathbf{A}_{1-3}+\mathbf{A}_{3-5}+\mathbf{A}_{3-4}$

$\mathbf{A}_{31}=\mathbf{A}_{1-3};\quad \mathbf{A}_{32}=\mathbf{A}_{23}=\mathbf{0};\quad \mathbf{A}_{34}=\mathbf{A}_{3-4};\quad \mathbf{A}_{35}=\mathbf{A}_{3-5}$

Etc.

De exemplu, explicit:

$$\mathbf{A}_{1-3} = \begin{bmatrix} A_{1-3}^{11} & A_{1-3}^{21} \\ A_{1-3}^{12} & A_{1-3}^{22} \end{bmatrix},$$

în care:

$$A_{1-3}^{LM} = \frac{1}{\Delta s_{1-3}^0}[\delta_{LM} T_{1-3}^0 + (\lambda_{1-3}^0 A_{1-3}^0 Y_{1-3}^0 - T_{1-3}^0)\frac{V_{1-3}^L V_{1-3}^M}{|\mathbf{V}_{1-3}|^3}]; \qquad L,M = 1,2 \quad (L,M = x,y)$$

■

Exemplu numeric

Ferma-cablu, cu:

- Deschidere: $2l = 42m$ (7 panouri × 6m)
- Săgeţi: $f_s = 3.5m$(superior); $f_i = 2.1m$ (inferior)
- Înălţime: $h = 7.6m$
- Modul de elasticitate: $Y = 16500kN/cm^2$
- Arii: $A_s = 20cm^2$; $A_i = 12cm^2$; $A_m = 0.6cm^2$
- Pretensionare – cablu inferior: $H_i^0 = 318kN$
- Incărcare: $P = 10kN$, vertical în jos, în nodurile superioare.

Matricea de rigiditate – în configuraţia de echilibru (rezultat NELSAS)

Este listat numai triunghiul superior al matricii (matrice simetrică).

```
 Matricea de rigiditate (Triunghiul superior) (kN/m); Semi-latime de
banda: 6

     101011.7        -24055.02        -3.328782         1.270904
-52142.03         9995.621
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000

     8098.644         1.270904        -2041.976         9995.621
-1963.201         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
```

```
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000

     64100.52         9009.456         0.000000         0.000000
-32427.00        -3653.484
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000         0.000000         0.000000

     3457.006         0.000000         0.000000        -3653.484
-459.8479         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000         0.000000

     106432.1        -15202.19        -5.326690         3.293367
-54284.75         5203.277
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000

     5782.588         3.293367        -3274.865         5203.277
-544.5213         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000

     65326.99         5502.978         0.000000         0.000000
-32894.66        -1852.787
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000

     3886.843         0.000000         0.000000        -1852.787
-152.1300         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000         0.000000         0.000000
0.000000

     109324.2        -5205.998        -7.625033         2.721410
-55031.80        1.7745042E-06
     0.000000         0.000000         0.000000         0.000000
0.000000         0.000000
     0.000000         0.000000         0.000000         0.000000
```

```
     5280.904          2.721410         -4691.024         1.7745042E-06
-45.35775          0.000000
     0.000000          0.000000          0.000000          0.000000
0.000000          0.000000
     0.000000          0.000000          0.000000

     65955.29          1850.065          0.000000          0.000000
-33053.00         1.0778679E-06
     0.000000          0.000000          0.000000          0.000000
0.000000          0.000000
     0.000000          0.000000

     4890.779          0.000000          0.000000         1.0778679E-06
-47.62442          0.000000
     0.000000          0.000000          0.000000          0.000000
0.000000          0.000000
     0.000000

     109324.2          5205.998         -7.625033         -2.721411
-54284.75         -5203.277
     0.000000          0.000000          0.000000          0.000000
0.000000          0.000000

     5280.904         -2.721411         -4691.024         -5203.277
-544.5213          0.000000
     0.000000          0.000000          0.000000          0.000000
0.000000

     65955.29         -1850.065          0.000000          0.000000
-32894.66          1852.787
     0.000000          0.000000          0.000000          0.000000

     4890.779          0.000000          0.000000          1852.787
-152.1300          0.000000
     0.000000          0.000000          0.000000

     106432.1          15202.19         -5.326690         -3.293367
-52142.03         -9995.621
     0.000000          0.000000

     5782.588         -3.293367         -3274.865         -9995.621
-1963.201          0.000000
     0.000000

     65326.99         -5502.978          0.000000          0.000000
-32427.00          3653.484

     3886.843          0.000000          0.000000          3653.484
-459.8479

     101011.7          24055.02         -3.328782         -1.270904

     8098.644         -1.270904         -2041.976

     64100.52         -9009.456

     3457.006
```

Verificare – pentru elementul diagonal A_{55}:

Linia 5 a matricii (corespunzând la ΔU_{3x}), completată cu elementele 51 ... 54 (egale cu elementele 15 ... 45) este:

```
-52142.03     -1963.201          -32427.00         -459.8479
106432.1        -15202.19          -5.326690          3.293367
-54284.75         5203.277
```

Suma elementelor non-diagonale corespunzând lui ΔU_{1x}, ΔU_{4x} şi ΔU_{5x} (nodurile conectate cu 3 sunt 1, 4, 5), luate cu semn schimbat, este:

52142.03 + 5.326690 + 54284.75 = 106432.10669

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